Partial Differentiation

Let $u = f(x, y)$.

Then $\dfrac{\partial u}{\partial x} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x, y) - f(x, y)}{\delta x}$

$\dfrac{\partial u}{\partial y} = \lim\limits_{\delta y \to 0} \dfrac{f(x, y+\delta y) - f(x, y)}{\delta y}$

Exp 1. If $u = x^2 + y^2 + z^2$

Then $\dfrac{\partial u}{\partial x} = 2x + 0 + 0 = 2x$

$\dfrac{\partial u}{\partial y} = 0 + 2y + 0 = 2y$

& $\dfrac{\partial u}{\partial z} = 0 + 0 + 2z = 2z$.

2. If $u = f(x, y, z)$

Let $u = x^3y + 2xy^2 + yz^3 + xz$

Then $\dfrac{\partial u}{\partial x} = 3x^2y + 2\cdot 1\cdot y^2 + 0 + 1\cdot z = 3x^2y + 2y^2 + z$.

$\dfrac{\partial u}{\partial y} = x^3\cdot 1 + 2x\cdot 2y + 1\cdot z^3 + 0$

$= x^3 + 4xy + z^3$

$\dfrac{\partial u}{\partial z} = 0 + 0 + y\cdot 3z^2 + x\cdot 1 = 3yz^2 + x$.

3. If $u = x^3y + x^2y^2 + xy^3$

then show that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$

A:- $u = x^3y + x^2y^2 + xy^3$

$\dfrac{\partial u}{\partial y} = x^3\cdot 1 + x^2\cdot 2y + x\cdot 3y^2 = x^3 + 2x^2y + 3xy^2$

$\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = 3x^2 + 4xy + 3\cdot 1\cdot y^2 = 3x^2 + 4xy + 3y^2$ —— (1)

Now $\dfrac{\partial u}{\partial x} = 3x^2y + 2xy^2 + y^3$

$\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = 3x^2\cdot 1 + 4x\cdot y + 3y^2 = 3x^2 + 4xy + 3y^2$ —— (2)

From (1) & (2); $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$. Proved

Note: If $u = f(x,y)$, then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Homogeneous Function: If the degree of each and every term of a function are same, then the function is known as Homogeneous Function:

Exp: $f(x,y) = x^3 + x^2 y + xy^2 + y^3 \longrightarrow$ Homogeneous

but $f(x,y) = x^3 + 3xy + 2xy^2 + y^4 \longrightarrow$ Non-homogeneous.

Euler's Theorem on Homogeneous Function:- If u is a homogeneous function of x & y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

proof:- since u is a homogeneous function of x & y of degree n, $\therefore$ suppose $u = x^n f\left(\frac{y}{x}\right)$ ——— (i)

$u = x^n f(v)$ where $v = \frac{y}{x}$

$$\therefore \frac{\partial u}{\partial x} = x^n f'(v) \cdot -\frac{y}{x^2} + n x^{n-1} f(v)$$

$$= -y x^{n-2} f'(v) + n x^{n-1} f(v)$$

$$\therefore x\frac{\partial u}{\partial x} = -xy x^{n-2} f'(v) + n \cdot x \cdot x^{n-1} f(v)$$

$$x\frac{\partial u}{\partial x} = -y x^{n-1} f'(v) + n x^n f(v) \quad \text{——— (2)}$$

Again differentiating (i) Partially w.r to y, we get

$$\frac{\partial u}{\partial y} = x^n f'(v) \cdot \frac{1}{x}$$

$$= x^{n-1} f'(v)$$

$$\therefore y\frac{\partial u}{\partial y} = y x^{n-1} f'(v) \quad \text{——— (3)}$$

Adding (2) & (3); we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{-y x^{n-1} f'(v)} + n x^n f(v) + \cancel{y x^{n-1} f'(v)}$$

$$= n x^n f(v)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \textbf{proved.}$$

Note: If $u = f(x,y)$, $\frac{\partial u}{\partial x}$ or u_x | $\frac{\partial^2 u}{\partial x^2} = u_{xx}$

$\frac{\partial u}{\partial y}$ or u_y | $\frac{\partial^2 u}{\partial y^2} = u_{yy}$

$\frac{\partial^2 u}{\partial x \partial y} = u_{xy}$;

Problems :

1. If $u = \log(x^2+y^2)$; show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Proof:- $u = \log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{(x^2+y^2)\cdot 2 - 2x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

Now $\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 2 - 2y \cdot 2y}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2(y^2-x^2+x^2-y^2)}{(x^2+y^2)^2} = 0 \quad \text{Proved.}$$

2. If $u = \sqrt{x^2+y^2+z^2}$; prove that $u_{xx}+u_{yy}+u_{zz} = \frac{2}{u}$.

Proof:- $u = \sqrt{x^2+y^2+z^2}$

$$u_x = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x = \frac{x}{u}$$

$$\therefore u_{xx} = \frac{u\cdot 1 - x\cdot u_x}{u^2}$$

$$= \frac{u - x\cdot \frac{x}{u}}{u^2} = \frac{u^2-x^2}{u^3}$$

Similarly we can get $u_{yy} = \frac{u^2-y^2}{u^3}$ & $u_{zz} = \frac{u^2-z^2}{u^3}$

$$\therefore u_{xx}+u_{yy}+u_{zz} = \frac{3u^2-(x^2+y^2+z^2)}{u^3}$$

$$= \frac{3u^2-u^2}{u^3} = \frac{2u^2}{u^3}$$

$$= \frac{2}{u}. \quad \text{Proved}$$

3. If $u = x \sin^{-1}\left(\frac{y}{x}\right)$; then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

Ans:- $u = x\sin^{-1}\left(\frac{y}{x}\right)$

$\frac{u}{x} = \sin^{-1}\left(\frac{y}{x}\right)$

$\therefore \sin\left(\frac{u}{x}\right) = \frac{y}{x} = yx^{-1}$ ——— (1)

Differentiating (1) Partially both sides w r to x;

$\cos\left(\frac{u}{x}\right)\cdot\left[\frac{x\cdot\frac{\partial u}{\partial x} - u\cdot 1}{x^2}\right] = -y\cdot x^{-2} = -\frac{y}{x^2}$

$\cos\left(\frac{u}{x}\right)\cdot \frac{x\frac{\partial u}{\partial x} - u}{\cancel{x^2}} = \frac{-y}{\cancel{x^2}}$

$\therefore x\frac{\partial u}{\partial x} - u = -y\cdot\frac{1}{\cos(u/x)} = -y\sec\frac{u}{x}$.

$x\frac{\partial u}{\partial x} = -y\sec\frac{u}{x} + u$ ——— (2)

Again differentiating Partially (1) w. r. to y,

$\cos\left(\frac{u}{x}\right)\cdot\left[\frac{1}{x}\frac{\partial u}{\partial y}\right] = 1\cdot x^{-1}$

$\cos\left(\frac{u}{x}\right)\left[\frac{1}{\cancel{x}}\frac{\partial u}{\partial y}\right] = \frac{1}{\cancel{x}}$

$\therefore \frac{\partial u}{\partial y} = \frac{1}{\cos(u/x)} = \sec(u/x)$

$y\frac{\partial u}{\partial y} = y\sec\left(\frac{u}{x}\right)$ ——— (3)

Adding (2) & (3); we get

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{-y\sec\frac{u}{x}} + u + \cancel{y\sec\left(\frac{u}{x}\right)}$

$= u$. Proved

4. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Ans:- $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$\therefore \tan u = \frac{x^3+y^3}{x+y}$

$$\log(\tan u) = \log\left(\frac{x^3+y^3}{x+y}\right) = \log(x^3+y^3) - \log(x+y) \quad \text{--- (1)}$$

Differentiating partially both sides of (1) w.r. to x,

$$\frac{1}{\tan u}\cdot \sec^2 u \cdot \frac{\partial u}{\partial x} = \frac{1}{x^3+y^3}\cdot 3x^2 - \frac{1}{x+y}\cdot 1$$

$$\frac{1}{\sin u \cos u}\cdot\frac{\partial u}{\partial x} = \frac{3x^2}{x^3+y^3} - \frac{1}{x+y}$$

$$\frac{\partial u}{\partial x} = \sin u \cos u \cdot \left[\frac{3x^2}{x^3+y^3} - \frac{1}{x+y}\right]$$

$$\therefore x\frac{\partial u}{\partial x} = \sin u \cos u \left[\frac{3x^3}{x^3+y^3} - \frac{x}{x+y}\right] \quad \text{--- (2)}$$

Similarly we differentiate partially (1) w.r to y, we can get,

$$y\frac{\partial u}{\partial y} = \sin u \cos u \left[\frac{3y^3}{x^3+y^3} - \frac{y}{x+y}\right] \quad \text{--- (3)}$$

Adding (2) & (3);

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin u \cos u \left[\frac{3(x^3+y^3)}{x^3+y^3} - \frac{x+y}{x+y}\right]$$

$$= \sin u \cos u \left[3-1\right]$$

$$= 2\sin u \cos u$$

$$= \sin 2u. \text{ proved.}$$

5. If $u = \log(x^3+y^3+z^3-3xyz)$; then show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Ans:-

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \quad \text{—— (1)}$$

Now $u = \log(x^3+y^3+z^3-3xyz)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\cdot(3x^2-3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}\cdot(3y^2-3xz)$$

& $$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\,\cancel{(x^2+y^2+z^2-yz-zx-xy)}}{(x+y+z)\,\cancel{(x^2+y^2+z^2-yz-zx-xy)}}$$

$$= \frac{3}{(x+y+z)}$$

From (1); $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= 3\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right](x+y+z)^{-1}$$

$$= 3\left[-\frac{1\cdot 1}{(x+y+z)^2}-\frac{1\cdot 1}{(x+y+z)^2}-\frac{1\cdot 1}{(x+y+z)^2}\right]$$

$$= 3\left[\frac{-3}{(x+y+z)^2}\right]$$

$$= \frac{-9}{(x+y+z)^2}\cdot \text{ proved.}$$